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Invariants and Chaotic Maps

W.-H. Steeb[1,2] and M. A. van Wyk[1]

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A one-dimensional map $f(x)$ is called an invariant of a two-dimensional map $g(x, y)$ if $g(x, f(x)) = f(f(x))$. The logistic map is an invariant of a class of two-dimensional maps. We construct a class of two-dimensional maps which admit the logistic maps as their invariant. Moreover, we calculate their Lyapunov exponents. We show that the two-dimensional map can show hyperchaotic behavior.

The logistic equation

$$x_{t+1} = 2x_t^2 - 1, \quad t = 0, 1, 2, \ldots, \quad x_0 \in [-1, 1] \tag{1}$$

is the most studied equation with chaotic behavior (Steeb, 1992, 1993, 1994). All quantities of interest in chaotic dynamics can be calculated exactly. Examples are the fixed points and their stability, the periodic orbits and their stability, the moments, the invariant density, the topological entropy, the metric entropy, the Lyapunov exponent, and the autocorrelation function. The exact solution of (1) takes the form

$$x_t = \cos(2^t \arccos(x_0)) \tag{2}$$

since $\cos(2\alpha) = 2\cos^2(\alpha) - 1$. The Lyapunov exponent for almost all initial conditions is given by $\ln(2)$. The logistic equation is an invariant of a class of second-order difference equations

$$x_{t+2} = g(x_t, x_{t+1}), \qquad t = 0, 1, 2, \ldots \tag{3}$$

This means that if (1) is satisfied for a pair (x_t, x_{t+1}), then (3) implies that (x_{t+1}, x_{t+2}) also satisfies (1). In other words, let

$$x_{t+1} = f(x_t), \qquad t = 0, 1, 2, \ldots \tag{4}$$

[1] Department of Applied Mathematics and Nonlinear Studies, Rand Afrikaans University, Auckland Park 2006, South Africa; e-mail: WHS@RAU3.RAU.AC.ZA
[2] National University of Singapore, Computational Science Programme, Republic of Singapore.

be a first-order difference equation. Then (4) is called an invariant of (3) if

$$g(x, f(x)) = f(f(x)) \tag{5}$$

The second-order difference equation (3) can be written as a first-order system of difference equations ($x_{1,t} \equiv x_t$)

$$x_{1,t+1} = x_{2,t}, \qquad x_{2,t+1} = g(x_{1,t}, x_{2,t}) \tag{6}$$

If x_0 and x_1 are the initial conditions of (3) ($x_0, x_1 \in [-1, 1]$) and assuming that (1) is an invariant of (3) as well as that x_0 and x_1 satisfy the logistic equation, then a one-dimensional Lyapunov exponent of (6) is given by ln(2). Since system (6) is two-dimensional, we have a second one-dimensional Lyapunov exponent and a two-dimensional Lyapunov exponent. Let λ_1^{I} and λ_2^{I} be the two one-dimensional Lyapunov exponents. Let λ^{II} be the two-dimensional Lyapunov exponent. Then we have

$$\lambda^{\mathrm{II}} = \lambda_1^{\mathrm{I}} + \lambda_2^{\mathrm{I}} \tag{7}$$

Let us find the two-dimensional Lyapunov exponent. Consider the system of first-order difference equations

$$x_{1,t+1} = f_1(x_{1,t}, x_{2,t}), \qquad x_{2,t+1} = f_2(x_{1,t}, x_{2,t}) \tag{8}$$

The variational equation is given by $[\mathbf{x}_t = (x_{1,t}, x_{2,t})]$

$$\begin{aligned} y_{1,t+1} &= \frac{\partial f_1}{\partial x_1}(\mathbf{x}_t) y_{1,t} + \frac{\partial f_1}{\partial x_2}(\mathbf{x}_t) y_{2,t} \\ y_{2,t+1} &= \frac{\partial f_2}{\partial x_1}(\mathbf{x}_t) y_{1,t} + \frac{\partial f_2}{\partial x_2}(\mathbf{x}_t) y_{2,t} \end{aligned} \tag{9}$$

Let $\mathbf{y}_t$ and $\mathbf{v}_t$ be two quantities satisfying the variational equation (9). Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be two unit vectors in $\mathcal{R}^2$ with $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$, where the dot denotes the scalar product. Let $\wedge$ be the exterior product [Grassmann product (Steeb, 1993, 1994)]. Then we find

$$\mathbf{y}_t \wedge \mathbf{v}_t = (y_{1,t} v_{2,t} - y_{2,t} v_{1,t}) \mathbf{e}_1 \wedge \mathbf{e}_2 \tag{10}$$

Now we define

$$w_t := y_{1,t} v_{2,t} - y_{2,t} v_{1,t} \tag{11}$$

Thus the time evolution of w_t is given by

$$w_{t+1} = \left(\frac{\partial f_1}{\partial x_1}(\mathbf{x}_t) \frac{\partial f_2}{\partial x_2}(\mathbf{x}_t) - \frac{\partial f_1}{\partial x_2}(\mathbf{x}_t) \frac{\partial f_2}{\partial x_1}(\mathbf{x}_t) \right) w_t \tag{12}$$

The two-dimensional Lyapunov exponent is given by

$$\lambda^{II} = \lim_{T\to\infty} \frac{1}{T} \ln|w_T| \tag{13}$$

Obviously, λ_1^I, λ_2^I, and λ_{II} depend on the initial conditions of (8). If $f_1(x_1, x_2) = x_2$ and $f_2(x_1, x_2) = g(x_1, x_2)$ as in (6), we obtain from (12) that

$$w_{t+1} = -\frac{\partial g}{\partial x_1}(\mathbf{x}_t)w_t \tag{14}$$

Without loss of generality we can set $w_0 = 1$.

We derive now a class of second-order difference equation with the logistic map as an invariant. Our ansatz for $g(x_1, x_2)$ with $f(x) = 2x^2 - 1$ is given by

$$g(x_1, x_2) = a_{10}x_1 + a_{01}x_2 + a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 + d \tag{15}$$

Satisfying the condition (5) yields

$$g(x_1, x_2) = x_2 - 2x_1^2 + 2x_2^2 + d(1 + x_2 - 2x_1^2) \tag{16}$$

Since

$$\frac{\partial g}{\partial x_1} = -4x_1(d + 1) \tag{17}$$

we find that (14) takes the form

$$w_{t+1} = -4x_{1,t}(d + 1)w_t \tag{18}$$

Let us now calculate the two-dimensional Lyapunov exponent λ^{II}. The initial values $x_{1,0}$, $x_{2,0}$ of the two-dimensional map $x_{1,t+1} = x_{2,t}$, $x_{2,t+1} = g(x_{1,t}, x_{2,t})$ satisfy the logistic map in our following calculations. Using (2), (18), and (19), we obtain

$$\lambda^{II}(\theta_0) = \lim_{T\to\infty} \frac{1}{T} \ln\left(\prod_{t=1}^{T} 4|d + 1|\cdot|\cos(2^t\theta_0)|\right)$$
$$d \neq -1, \qquad \theta_0 := \arccos(x_0) \tag{19}$$

or

$$\lambda^{II}(\theta_0) = 2\ln 2 + \ln|d + 1| + \gamma(\theta_0) \tag{20}$$

where

$$\gamma(\theta_0) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \ln|\cos(2^t\theta_0)| \tag{21}$$

Now, since

$$\cos(2^t\theta_0) = \cos(2^t\theta_0 \bmod 2\pi) \tag{22}$$

we only need to study the Bernoulli shift map

$$\theta_{t+1} = 2\theta_t \quad \bmod 2\pi \tag{23}$$

This map has the solution

$$\theta_t = 2^t\theta_0 \quad \bmod 2\pi \tag{24}$$

The map (23) is ergodic with the invariant density

$$\rho(\theta) = \frac{1}{2\pi}\chi_{[0,2\pi)}(\theta) \tag{25}$$

where χ is the characteristic function. Thus we may apply Birkhoff's ergodic theorem (Steeb, 1992). This then gives

$$\begin{aligned}\gamma(\theta_0) &= \int_0^{2\pi} \rho(\theta)\ln|\cos\theta|\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \ln|\cos\theta|\,d\theta \\ &= \frac{2}{\pi}\int_0^{\pi/2} \ln(\cos\theta)\,d\theta \end{aligned} \tag{26}$$

It follows that

$$\gamma(\theta_0) = -\ln 2 \qquad \text{for a.e. } \theta_0 \in [0, 2\pi) \tag{27}$$

Thus

$$\lambda^{\mathrm{II}} = \ln 2 + \ln|d + 1|, \qquad d \neq -1 \tag{28}$$

Now, since one of the one-dimensional Lyapunov exponent is ln 2, and

$$\lambda^{\mathrm{II}} = \lambda_1^{\mathrm{I}} + \lambda_2^{\mathrm{I}}, \qquad \lambda_1^{\mathrm{I}} \geq \lambda_2^{\mathrm{I}} \tag{29}$$

we find the two one-dimensional Lyapunov exponent as

$$\lambda_1^{\mathrm{I}} = \max\{\ln 2, \ln|d + 1|\} \tag{30}$$

$$\lambda_2^{\mathrm{I}} = \min\{\ln 2, \ln|d + 1|\} \tag{31}$$

Obviously λ^{II} can be made arbitrarily large positive or negative by appropriate choice of d. This implies that the spectrum of the one-dimensional Lyapunov exponents may be $(+, -)$, $(+, 0)$, or $(+, +)$. Thus hyperchaos can occur.

Now, let $\{x_n(x_0)\}$ denote the orbit originating from x_0 for the logistic map (1). Then

$$\{x_n(x_0)\} \text{ is chaotic} \Leftrightarrow \arccos(x_0) \in \mathcal{R}\backslash\mathcal{Q} \qquad (32)$$

This follows from the fact that the orbit of the Bernoulli shift map is chaotic if and only if $\theta_0 \in \mathcal{R}\backslash\mathcal{Q}$.

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